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(Translation)

ABOUT THE HARMONIC QUADRILATERAL

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Abstract.

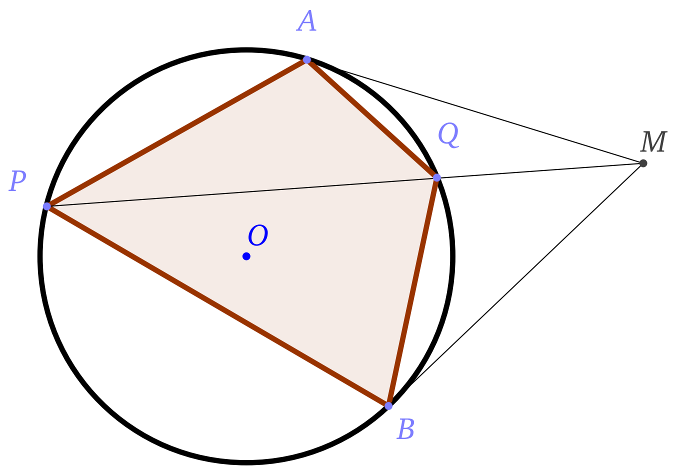
Exploring the world of geometry, we encounter a special quadrilateral with unique properties, known as the beautiful quadrilateral. Since the problems in this topic largely concerned with the idea of harmonic points, a more appropriate term is harmonic quadrilateral. In this small article, we represent some fundamental properties of harmonic quadrilateral and their application in solving geometry problems.

1 Fundamental blocks

Definition 1.1. A cyclic quadrilateral for which the product of the opposite sides is the same, is called a *harmonic quadrilateral*. In a particular case, a cyclic quadrilateral $ABCD$ such that $\frac{AB}{AD} = \frac{CB}{CD}$ is a *harmonic quadrilateral.*

Example 1.1. Given a circle ω and a point M outside it. Let MA and MB be tangents to the circle at A and B. Let l be an arbitrary line through M which intersects the circle at P and Q. Then $APBQ$ is a harmonic quadrilateral.

Proof.



It suffices to show that $\dfrac{AQ}{AP} = \dfrac{BQ}{BP}$. Since we have ΔMAQ is similar to ΔMPA,

$$\frac{AQ}{AP} = \frac{MQ}{MP}.$$

Additionally, ΔMBQ is also similar to ΔMPB, so

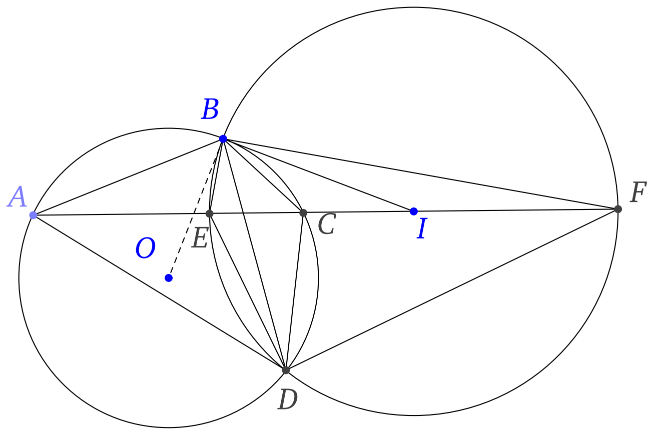
$$\frac{BQ}{BP} = \frac{MQ}{MP}.$$

Combining these, we obtain the desired result. This implies that the quadrilateral $AQBP$ is harmonic. ■

Comment: In the above example, we clearly observe that if a quadrilateral inscribed in a circle has its two tangents at two opposite vertices met at a point that lies on its diagonal, then the quadrilateral is harmonic. Naturally, we may wonder if the inversed statement is correct: every harmonic quadrilateral inscribed in a circle has its two tangents at two opposite vertices met at a point that lies on its diagonal. The answer will be revealed in later part.

Property 1.1. Let the harmonic quadrilateral $ABCD$ be inscribed the circle (O) and $\frac{AB}{BC} = \frac{DA}{DC} = k$. Show that (O) and a circle of Apollonius with ratio k are orthocentric.

Proof.



The internal and external bisector of $\angle ABC$ intersect the line AC at E and F, respectively. Note that the circle (I) with diameter EF is the circle of Apollonius with ratio k. Clearly, B and D lie on (I). We only need to prove that IB is tangent to the circle (O).

Indeed, we have $(ACED) = -1$ and I is the midpoint of EF then $IE^2 = IA \cdot IC$. Furthermore, since ΔIBE is the isosceles triangle with $IB = IE$, we deduce $IB^2 = IA \cdot IC$, which implies that IB is tangent to the circle (O). The proof is completed. ■

Property 1.2. The quadrilateral $ABCB$ inscribed the circle (O) is harmonic iff AC, the tangents to (O) at B and D are concurrent. (AC and BD are not diameters)

Proof. This property can be deduced from **Example 1.1.** and **Property 1.1.**. ■

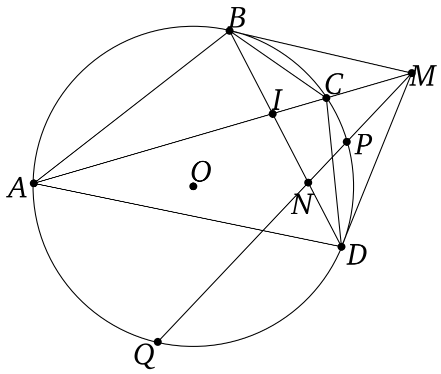
Property 1.3. In a harmonic quadrilateral $ABCD$, $AC \cdot BD = 2AB \cdot CD = 2BC \cdot AD$.

Proof. Since $ABCD$ is harmonic, $\frac{AB}{AD} = \frac{CB}{CD}$, which is equivalent to $AB \cdot CD = AD \cdot CB$.

On the other hand, the quadrilateral $ABCD$ is cyclic. Then, applying Ptolemy's theorem, we get $AC \cdot BD = AB \cdot CD + AD \cdot BC$. Hence, $AC \cdot BD = 2AB \cdot CD = 2BC \cdot AD$. ■

Property 1.4. Let $ABCD$ be a harmonic quadrilateral inscribed in the circle ω. The tangents to the circle at B and D intersect at M. Denote by I the intersection of AC and BD. Then we have $(MIAC) = -1$.

Proof.



We have

$$\frac{MC}{MA} = \frac{S_{MCD}}{S_{MAD}} = \frac{CD \cdot \sin \angle MDC}{AD \cdot \sin \angle MDA},$$

and

$$\frac{IC}{IA} = \frac{S_{ICB}}{S_{IAB}} = \frac{BC \cdot \sin \angle IBC}{BA \cdot \sin \angle IBA}.$$

Moreover,

$$\frac{CD}{AD} = \frac{BC}{AB}, \quad \sin \angle ABI = \sin \angle ADM, \quad \sin \angle IBC = \sin \angle MDC.$$

Combining these, we get $\frac{MC}{MA} = \frac{IC}{IA}$, which follows that $(MICA) = -1$. The proof is completed. ■

Comment: Let a point M outside the circle (O). Draw 2 tangents MA, MB to (O). A secant through M intersects (O) at P, Q and cuts AB at N. Then $(MNPQ) = -1$.

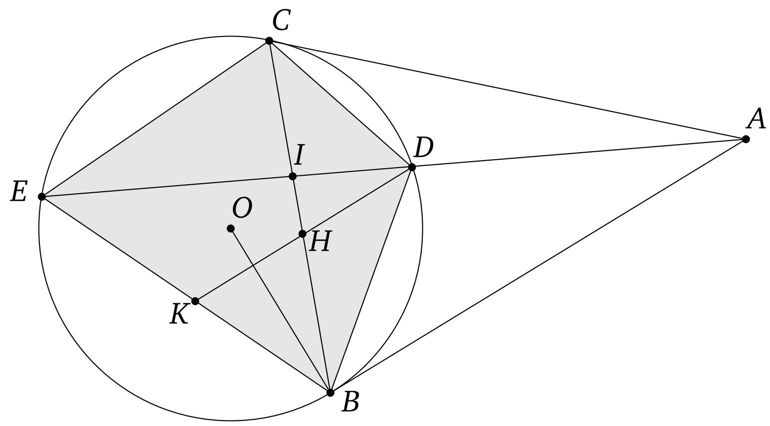
Property 1.5. Let $ABCD$ be a harmonic quadrilateral inscribed in the circle ω which has center O. The tangents to the circle at B and D intersect at M. Denote by I the intersection of OM and BD. Then IB is an angle bisector of $\angle AIC$.

Proof. Line AC meets line BD at K. Then $(MKAC) = -1$. We have $I(MKAC) = -1$ and IM is perpendicular to IK, hence IM and IK are the internal and external bisectors of $\angle AKC$, respectively. The proof is completed. ■

The **Property 1.1.** and **Property 1.2.** show another significant methods to prove that *3 points are collinear* or *3 lines are concurrent*. The **Property 1.3.** is beautiful and the **1.4.** is the reason why this quadrilateral is called *harmonic quadrilateral.*

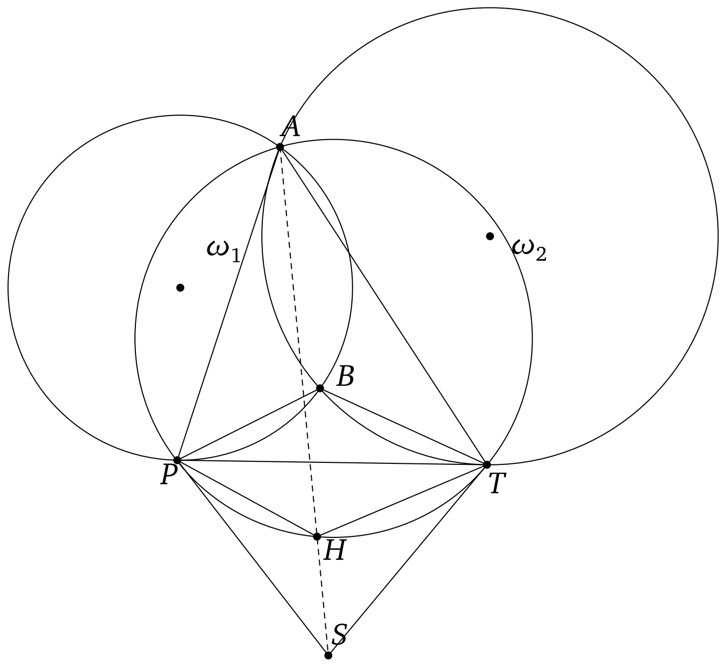
2 Applications

Problem 2.1. Consider a circle (O) and an arbitrary point A outside it. Let AB and AC be tangents from A to (O) and l be an arbitrary line through A intersecting the circle at D and E (so that A, D and E are collinear in this order). Consider a line l' through D which is perpendicular to OB, and intesects BC, BE at H, K respectively. Prove that $DH = HK$.
Proof.



The line BC intersects the line ED on the point I. Thus, we get that the quadrilateral $ECDB$ is harmonic, so $B(EDIA) = -1$. Furthermore, $DK \parallel AB$ since they are perpendicular to OB. Hence, H is the midpoint of the line DK, we conclude that $HD = HK$, as desired. ■

Problem 2.2. *[Vietnam TST 2001]* Given two circles ω_1, ω_2 intersecting at A and B. Let l be any one of those common circles which tangents to ω_1, ω_2 at P, T respectively. The tangents at P and T to the circle inscribing $\triangle APT$ intersect at S. Denote by H the reflection of the point B in the sideline PT. Prove that A, H and S are collinear.
Proof.



It suffices to show that $APHT$ is harmonic.

First, we prove that $APHT$ is cyclic. Indeed, we have $\angle BPT = \angle PAB$ and $\angle BTP = \angle BAT$, which leads to $\angle BPT + \angle BTP = \angle PAT$ or $\angle PAT = \frac{\pi}{2} - \angle PBT = \frac{\pi}{2} - \angle PHT$. Thus, the quadrilateral $APHT$ is cyclic.

Now denote by M the intersection of AB and PT. We obtain that

$$MB \cdot MA = MP^2 \quad \text{and} \quad MB \cdot MA = MT^2,$$

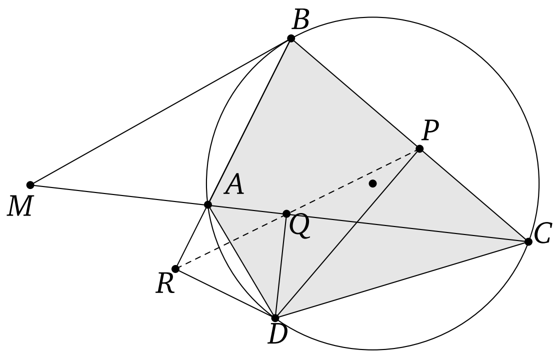
which yield $MP = MT$.

Since ΔMPB is similar to ΔMAP, $\frac{BP}{AP} = \frac{MB}{MP}$. In the same manner, $\frac{BT}{AT} = \frac{MB}{MT}$. Hence, $\frac{BP}{AP} = \frac{BT}{AT}$, which is $\frac{BP}{BT} = \frac{AP}{AT}$.

Furthermore, we have $BP = HP$ and $BT = HT$. Then $\frac{AP}{AT} = \frac{HP}{HT}$. For this reason, the quadrilateral $APHT$ is harmonic. From **Property 1.2.**, the claim follows which ends the proof. ■

Problem 2.3. Let $ABCD$ be a cyclic quadrilateral. Denote by P, Q and R the orthogonal projections of the point D on the lines BC, CA and AB respectively. Prove that $PQ = QR$ iff the angle bisectors of $\angle ABC$ and $\angle ADC$ intersect at the point lies on AC.

Proof.



The 3 points P, Q, R are collinear, which follows from Simson's Line.

Let d be the line which passes through point B and is parallel to PR. d cuts the line AC at point M. The angle bisectors of $\angle ABC$ and $\angle ADC$ converge at the point on AC if and only if $\frac{BA}{BC} = \frac{DA}{DC}$. Moreover, $QP = QR$ is equivalent to $(MQAC) = -1$. For these reasons, we need to show that $ABCD$ is harmonic iff $(MQAC) = -1$.

To verify this, we construct some equalities

$$\frac{AM}{AQ} = \frac{AB}{AR} \quad \text{and} \quad \frac{CM}{CQ} = \frac{CB}{CP}.$$

On the other hand, since the triangles DAR and DCP are similar, we obtain $\frac{AR}{CP} = \frac{DA}{DC}$.

Combining these, we get $(MQAC) = -1$ iff

$$\frac{AB}{AR} = \frac{AM}{AQ} = \frac{CM}{CQ} = \frac{CB}{CP}$$

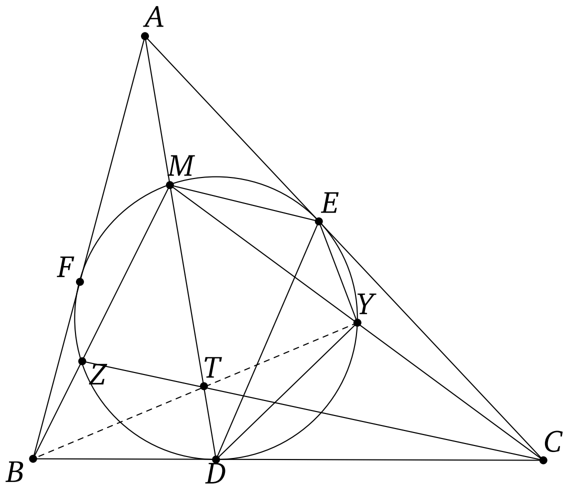
which is equivalent to

$$\frac{AB}{CB} = \frac{AR}{CP} = \frac{DA}{DC}.$$

The proof is completed. ■

Problem 2.4. The incircle ω of ΔABC has center I and touches BC, CA, AB at D, E, F respectively. Let M be the intersection of AD and ω ($M \neq D$). Denote by Y, Z the intesections of ω and BM, CM respectively. Prove that BZ, CY and AD are concurrent.

Proof.



Consider 2 harmonic quadrilaterals $MEYD$ and $MFZD$, we have

$$MY \cdot DE = 2ME \cdot DY, \text{ so } \frac{MY}{YC} = \frac{2ME \cdot DY}{YC \cdot DE},$$

and

$$MZ \cdot DF = 2MF \cdot DZ, \text{ so } \frac{BZ}{MZ} = \frac{BZ \cdot DF}{2MF \cdot DZ}.$$

Thus,

$$\frac{DC \cdot YM \cdot ZB}{DB \cdot YC \cdot ZM} = \left(\frac{DC}{DB}\right) \cdot \left(\frac{2ME \cdot DY}{YC \cdot DE}\right) \cdot \left(\frac{BZ \cdot DF}{2MF \cdot DZ}\right), \quad (1).$$

Additionally, $MEDF$ is harmonic, so $\frac{ME}{MF} = \frac{DE}{DF}$. For this reason, (1) is equivalent to

$$\frac{DC \cdot YM \cdot ZB}{DB \cdot YC \cdot ZM} = \left(\frac{DC}{DB}\right) \cdot \left(\frac{DY}{DZ}\right) \cdot \left(\frac{BZ}{CY}\right).$$

On the other hand, since $\frac{CY}{DY} = \frac{CD}{DM}$ and $\frac{BZ}{BD} = \frac{DZ}{DM}$, we have

$$\left(\frac{DC}{DB}\right) \cdot \left(\frac{DY}{DZ}\right) \cdot \left(\frac{BZ}{CY}\right) = 1.$$

Finally,

$$\frac{DC \cdot YM \cdot ZB}{DB \cdot YC \cdot ZM} = 1,$$

from Ceva's theorem, 3 lines MD, BY and CZ are concurrent, which ends the proof. ■

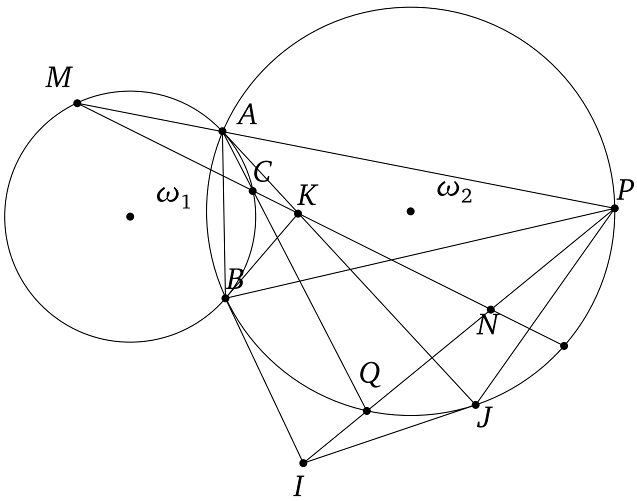
From this problem, we claim that EF, YZ, BC are concurrent and ZF, YE, AD have the same property. Indeed, if EF meets BC at K then $(KDBC) = -1$ and if YZ cuts BC at H then $(HDBC) = -1$ thanks to the concurrence of MD, BY and CZ. Thus, $K \equiv H$, as desired. Also, consider 2 triangles BZF and CYE, according to Desargues' theorem, ZF and EY meet at a point which lies on AM.

Problem 2.5. *[Vietnam TST 2001]* Given two circles ω_1, ω_2 intersecting at A and B. The tangents to ω_1 at A and B intersect at K. Denote by M the arbitrary point lies on ω_1 ($M \neq A$ and $M \neq B$). AM intersects ω_2 at P, KM intesects ω_1 at C, and AC intesects ω_2 at Q.

a) Prove that the midpoint of PQ lies on MC.

b) Prove that PQ passes through a fixed point when M varies on the circle ω_1.

Proof.



(a). Let MK intersect PQ at the point N, we will show that N is the midpoint of PQ.

We have $AMBC$ is harmonic, so $\frac{AC}{AM} = \frac{CB}{MB}$. We also have ΔMBP is similar to ΔCBQ, so $\frac{CQ}{MP} = \frac{CB}{MB}$. From these, we obtain $\frac{AC}{AM} = \frac{CQ}{MP}$.

Apply Menelaus's theorem for triangle APQ with the secant line MKN, we get

$$\left(\frac{NP}{NQ}\right) \cdot \left(\frac{MA}{MP}\right) \cdot \left(\frac{CQ}{CA}\right) = 1,$$

which leads us to $NP = NQ$ and implies that N is midpoint of PQ.

(b). AK cuts the circle ω_2 at a fixed point J. We will show that $BQJP$ is harmonic.

We have that $\angle CMB = \angle BAC = \angle BPQ$ and $\angle MBC = \angle CAP = \angle PBQ$, so the triangles CBM and QBP are similar. Hence, $\frac{BC}{BM} = \frac{BQ}{BP}$. Analogously, we also have $\frac{AC}{AM} = \frac{JQ}{JP}$. Additionally, $\frac{BC}{BM} = \frac{AC}{AM}$. Then $\frac{BQ}{BP} = \frac{JQ}{JP}$. Therefore, $BQJP$ is a harmonic quadrilateral. Thus, PQ passes through the intersection point I of two tangents at B and J to the circle ω_2. the proof is completed. ■

3 Problems

Problem 3.1. Consider a circle ω and an arbitrary point A outside it. Let AB and AC be tangents from A to ω and l_1, l_2 be arbitrary lines through A intersecting the at M, Q and N, P respectively (so that A, M, Q and A, N, Q are two collinear sets of points in those orders). Prove that BC, PM, QN are concurrent.

Problem 3.2. Let ABC be an isosceles triangle with $AB = AC$ and M be a midpoint of BC. Denote by P the point that satisfies $\angle ABP = \angle PCB$. Prove that $\angle BPM + \angle CPA = 180^o$.

Problem 3.3. Consider a circle (O) and a fixed point M outside it. Let MB be a tangent from M to (O) and l_1 be an arbitrary line through M intersecting ω at A and C (so that M, A and C are collinear in this order). Consider an arbitrary parallel line to MB that cuts BA and BC at N and P respectively. Denote by I a midpoint of NP. Prove that I belongs to a fixed line.

Problem 3.4. Let ABC be a triangle inscribed in a circle ω. An arbitrary line through A intersects ω at E and the tangents to the circle at B and C at M and N respectively. Prove that there exist a fixed point on EF.

Problem 3.5. Let $ABCD$ be cyclic quadrilateral. Denote by L and N be midpoints of AC and BD respectively. Prove that DB is an angle bisector of $\angle ANC$ iff AC is an angle bisector of $\angle BLD$.

4 References

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